

Einstein's theory of wavefronts versus Einstein's relativity of simultaneity

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Abstract

The relativity of simultaneity implies that the Lorentz transformed (LT) image of a spherical (circular) wavefront is not a spherical (circular) wavefront (Einstein 1905) but an ellipsoidal (elliptical) wavefront [Moreau W.]. We show first that the relativity of simultaneity leads to the consequence that the image of a Lorentz transformed plane wavefront is a tangent plane to an ellipsoid and not a tangent plane to a sphere (Einstein 1905). We then deduce a longitudinal component of the tangent vector to Poincaré's ellipse which is directly connected to the relativity of simultaneity. We suggest finally that this violation of simultaneity is related to Einstein's implicit choice of the (non relativistic) transverse gauge.

1 Relativity of simultaneity (LT) and Elliptical wavefront

Let us consider two inertial systems K and K' in uniform translation relative to one another. A source of light is at rest O in K. What is the image by a LT in K' in t_0 , to a circular wavefront in K, emitted at $t' = t = 0$ by this source S? Let us write the LT with perfectly spacetime symmetry (in x and t) [Poincaré H. 1905]:

$$x' = \gamma(x + \beta t) \quad y' = y \quad t' = \gamma(t + \beta x) \quad (1)$$

Let us consider the relativistic invariant:

$$x^2 + y^2 = r_0^2 = t_0^2 \quad x'^2 + y'^2 = r'^2 = t'^2 \quad (2)$$

The object time $t = t_0$ is fixed in K (circular wavefront in K) but the image time t' is not fixed (by the LT) in K'. We obtain using (1) $t' = \gamma^{-1}t_0 + \beta x'$:

$$x'^2 + y'^2 = (\gamma^{-1}t_0 + \beta x')^2 \quad \text{or} \quad (\gamma^{-1}x' - \beta t_0)^2 + y'^2 = t_0^2 \quad (3)$$

which is the Cartesian equation of an elongated ellipse in K' with the observer O' at the focus, **figure 1**):

The physical meaning of this elliptical spacetime wavefront is the **relativity of simultaneity**: two simultaneous events (*different abscissa*) in K are not simultaneous in K' [Pierseaux Y (2004)] :

$$\Delta t = 0 \quad \rightarrow \quad \Delta t' = \frac{t'^+ - t'^-}{2} \neq 0 \quad (4)$$

$\Delta t'$ is "the simultaneity gap" between two opposite events on the front (**figure 1**). If according to Einstein [Einstein A.(1905), paragraphe 3], the object (fixed time t) and the image (fixed time t'), are both spherical (circular)¹, then two simultaneous events in K must be always also simultaneous in K':

$$(\Delta t' = 0)_{\text{Einstein}} \quad (5)$$

¹Einstein writes : " At the time $t = \tau = 0$, when the origin of the two coordinates (K and k) is common to the two systems, let a **spherical wave** be emitted therefrom, and be propagated -with the velocity c in system K. If x, y, z be a point just attained by this wave, then $x^2 + y^2 + z^2 = c^2 t^2$. Transforming this equation with our equations of transformation (see Einstein's LT, 29), we obtain after a simple calculation $\xi^2 + \eta^2 + \zeta^2 = c^2 \tau^2$. The wave under consideration is therefore no less a **spherical wave** with velocity of propagation c when viewed in the moving system k." [Einstein A.(1905)]

In polar coordinates ($x' = r' \cos \theta'$, $y' = r' \sin \theta'$, with θ' as polar angle and F as pole) the equation of the elongated ellipse is :

$$r' = \frac{r_0}{\gamma(1 - \beta \cos \theta')} \quad (6)$$

With relativistic transformation of angle [Moreau W.]

$$\cos \theta' = \frac{\cos \theta + \beta}{1 + \beta \cos \theta} \quad (7)$$

We rediscover the LT for any point of the wavefront (r, θ or t, θ) :

$$r' = \gamma r_0 (1 + \beta \cos \theta) \quad t' = \gamma t_0 (1 + \beta \cos \theta) \quad (8)$$

The velocity of light ("one way") is identical in all directions within both systems because we have by construction **figure 1**): $\frac{r_0}{t_0} = \frac{r'^+}{t'^+} = \frac{r'^-}{t'^-} = c = 1$. What is a wavefront in a relativistic sense or in other words a "spacetime" wavefront? The invariant interval between the event "emission" ($t = t' = 0$) and any event on the wavefront ($t = t_0$, $r = r_0$ in K and t' , r' in K') is null within both systems. So Minkowski's null spacetime 4-vector is, from Poincaré's point of view², a wavefront 4-vector. *The elliptical image by a LT is deduced from a non-transversal t' section in Minkowski's cone.*

According to Einstein, the image of a space wavefront (t is fixed) is a space wavefront (t' is fixed). We will see now that is exactly the same for Einstein's plane wavefront [Einstein A.(1905), paragraphe 7].

2 Relativity of simultaneity (LT) and the tangent plane wavefront to the ellipsoid

Until now we considered only one source S_O which emits at $t = t' = 0$ a spherical wave at fixed time $t = t_0$ (figure 1)- Let us now consider a second source S_∞ at infinity at rest in K in the direction θ (**figure 2**) which emits a plane wavefront. Suppose that the considered plane wavefront (here a "wave straight line" at two space dimensions) be in O at the time $t = t' = 0$ (when S_O emits a spherical wave). It will be **tangent** in $t = t_0$ to the circular wavefront, emitted by S_1 (normalized $t_0 = r_0 = 1$ **figure 2**). We note that the simultaneous events $P_1 T P_2$ in K are no longer simultaneous ($P'_1 T' P'_2$) in K'. We deduce respectively (1) for the object front ("wave straight line" with angular coefficient $a = -\cot \theta$) and the image front the following relations:

$$x \cos \theta + y \sin \theta = t_0 \quad y' \sin \theta' + x' \cos \theta' = t' \quad (9)$$

And exactly as in the circular wavefronts case (2), there are two possibilities:

If t' is not fixed (Poincaré, spacetime image wavefront), we have by a LT, $t' = \gamma(t + \beta x) = \gamma^{-1}t + \beta x'$, and therefore the primed equation (9) is the equation of the tangent to the ellipse at the point T' (on which P'_1 and P'_2 are situated) :

$$y' \sin \theta' + x'(\cos \theta' - \beta) = \gamma^{-1}t \quad (10)$$

The angular coefficient of Poincaré's wavefront:

$$a'_{\text{poincaré}} = tg\alpha' = \frac{\beta - \cos \theta'}{\sin \theta'} \quad (11)$$

If t' is fixed (Einstein, space image wavefront³) the primed equation (9) is the equation of the tangent to the circle (with centre O' and radius r'_T). The angular coefficient of Einstein's wavefront is:

$$a'_{\text{einstein}} = -\cot \theta' \quad (12)$$

²Poincaré first introduced [Poincaré H.(1908)] the ellipsoidal image of a spherical wavefront. However Poincaré's historical ellipse (source at the focus) is not equation (6). It is given by the inverse LT by changing (in 6) the sign of the velocity and by inverting prime and non primed.

³Einstein defined clearly the image of the wavefront in paragraph 7 of his 1905 paper: "If we call the angle θ' the angle between the wave-normal (direction of the ray) and the direction of movement".

Both formulas are the same for $\theta' = 0$. **Einstein's double transversality** (or **Einstein's double simultaneity** $t_{fixed} \neq t'_{fixed}$) involves for the image front:

$$a' = -\cot \theta' \quad \Leftrightarrow \quad (\Delta t')_{front} = 0 \quad (13)$$

This is consistent with (5): Einstein's plane image wavefront is tangent to Einstein's spherical image wavefront. The phase Ψ of a sinusoidal monochromatic plane wave (\mathbf{A} is the amplitude) $\mathbf{A} = \mathbf{A}_0 \sin \Psi$ ($\mathbf{A}' = \mathbf{A}'_0 \sin \Psi'$) is defined by the 3-scalar product $\mathbf{k} \cdot \mathbf{r}$ ($\mathbf{k}' \cdot \mathbf{r}'$) with the frequency $\nu = \frac{\omega}{2\pi}$, the wave vector $\mathbf{k} = \frac{2\pi}{\lambda} \mathbf{1}_n$ with $\mathbf{1}_n$ the unit vector normal to the front ($\mathbf{k}' = \frac{2\pi}{\lambda'} \mathbf{1}_{n'}, \nu' = \frac{\omega'}{2\pi}$):

$$\omega t - \mathbf{k} \cdot \mathbf{r} = \Psi = \omega' t' - \mathbf{k}' \cdot \mathbf{r}' = \Psi' \quad \Leftrightarrow \quad \mathbf{A} \cdot \mathbf{k} = \mathbf{A}' \cdot \mathbf{k}' = 0 = Ak \cos \phi = A'k' \cos \phi' \quad (14)$$

This is a **Galilean invariant** (t is fixed on the object front and t' is fixed on the image front) in the sense that the angle (**figure 2**) $\phi = \phi' = 90^\circ$ is not altered by a Galilean transformation (GT).

$$\mathbf{k} \cdot \mathbf{r} \xrightarrow{GT} \mathbf{k}' \cdot \mathbf{r}' \quad \text{or} \quad \mathbf{A} \perp \mathbf{k} \xrightarrow{GT} \mathbf{A}' \perp \mathbf{k}' \quad (15)$$

But LT changes (11) this angle $\phi = 90^\circ$ into ϕ' in the following way (in **figure 2**, we note $\pi - \alpha'$):

$$\tan \phi' = \tan(\alpha' - \theta') = \frac{\beta \cos \theta' - 1}{\beta \sin \theta'} \quad (16)$$

The right angle $\phi = \phi' = 90^\circ$ is conserved *if and only if* the propagation is purely longitudinal ($\sin \theta' = 0$). Einstein considers [Einstein A.(1905), paragraphe 7] that the Galilean invariant $t_{fixed} = t'_{fixed}$ (14) is *by definition* a Lorentz invariant $t_{fixed} \neq t'_{fixed}$.⁴

3 Poincaré's longitudinal component of a vector tangent to the ellipse

Let us now consider a vector $\mathbf{T} = \overrightarrow{TP}$ on the tangent to the circle object. What is the image \mathbf{T}' by LT on the tangent to the ellipse of \mathbf{T} ?

$$x'_{T'} = \gamma(x_T + \beta t) \quad y'_{T'} = y_T \quad t'_{T'} = \gamma(t + \beta x_T) \quad x'_{P'} = \gamma(x_P + \beta t) \quad y'_{P'} = y_P \quad t'_{P'} = \gamma(t + \beta x_P) \quad (17)$$

We deduced the components $(T'_{x'}, T'_{y'})$ of the vector $\mathbf{T}' = \overrightarrow{T'P'}$ on the tangent to the ellipse (**figure 3**)

$$T'_{x'} = \Delta x' = \gamma \Delta x \quad T'_{y'} = \Delta y' = \Delta y \quad \Delta t' = \beta \gamma \Delta x \quad (18)$$

where $\Delta t'$ is "simultaneity gap".

Let us project the vectors \mathbf{T} and \mathbf{T}' on the direction of propagation and on the perpendicular to that direction in both systems K and K' (the origin is respectively in T and in T'). We deduce immediately from geometrical properties of the ellipse:

$$T = T_\perp = T'_\perp \quad T'_\parallel = \gamma \beta T_x = \Delta t' \quad (19)$$

Poincaré's theorem: "The simultaneity gap only depends on the longitudinal component T'_\parallel of the vector \mathbf{T}' of the tangent to the ellipse. Einstein cancels Poincaré's longitudinal component:

$$(T'_\parallel = \Delta t' = 0)_{\text{Einstein}} \quad (20)$$

Let us show now that we can associate to the spatial vector T and the Δt a 4-vector structure $(T_x, T_y, 0, \Delta t = 0) \rightarrow (T'_x, T'_y, 0, \Delta t')$ with invariant norm:

$$\|(T_x, T_y, 0, 0)\| = T^2 = T_\perp^2 \quad (21)$$

$$\|(T'_x, T'_y, 0, \Delta t')\| = T'^2 - \Delta t'^2 = T_\perp'^2 + T_\parallel'^2 - \Delta t'^2 = T_\perp'^2 \quad (22)$$

where the fourth component $\Delta t'$ is cancelled in Einstein's theory of wavefront.

⁴Everything happens as if $\beta = 0$ ", in other words exactly as in Einstein's synchronisation "in spherical waves" [Pierseaux Y.(ellipse)].

4 Conclusion

Einstein's theory of (spherical and plane) wavefronts is not compatible with Einstein's relativity of simultaneity. Einstein's wavefronts are "rigid" in a prerelativistic meaning: a set of simultaneous events is transformed into a set of simultaneous events. But this unexpected internal contradiction has possibly *no physical consequence*. Indeed we showed that only the electromagnetic potential (which is 4-vector written for the first time by Poincaré in 1905), and not the electromagnetic field, is sensitive to this violation of relativity of simultaneity. We showed [Pierseaux Y. (2005-2)][Pierseaux Y & Rousseaux G.] that Einstein's theory of wavefronts is based on the *transverse* gauge (completed Coulomb gauge) whilst Poincaré's theory of spacetime wavefronts is based upon the Lorenz gauge. The implicit Einstein's choice[Einstein A.(1905), paragraph 7] of the completed Coulomb gauge (transverse gauge) explains the reason why the relativity of simultaneity is violated.

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